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Uses of Think Aloud to Verify the Cognitive Attributes in Division of Fractions

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Abstract

The purpose of this technical report is to describe a research study exploring students' cognitive processing while solving problems related to division of fractions. Four students were interviewed using verbal protocol techniques with the intention of better understanding their cognitive processing when dividing fractions. Information gained from this study was used to verify and revise the cognitive model for division of fractions, which will be subsequently used to guide the development of a cognitive diagnostic test and instructional sequence.

Introduction

Ketterlin-Geller and colleagues proposed a cognitive model for division of fractions to explain students' cognitive processing when engaged in problem solving. The cognitive model was developed through evaluation of the mathematical rationale, input from mathematics education experts, and a comprehensive task analysis (see Ketterlin-Geller et al., in press, for details). Eleven attributes were identified and hierarchically organized that lead to mastery of the skills and knowledge needed to divide fractions. The resulting cognitive model is presented in Appendix A.

Verbal Protocol Techniques

Verbal protocols (also called Think-aloud protocols) require subjects to verbalize the cognitive processes involved in solving a problem. The purpose of the think-aloud strategy is to transform a problem-solver's covert thinking process into an overt and observable behavior, so that the thinking process can be documented and analyzed (Someren, Barnard & Sandberg, 1994). The interviewer asks the subject to reflect on his or her thinking process after the task is completed (retrospection); or to report on the thinking process while solving the problem (introspection). The interviewer uses questioning, prompting or dialogues to encourage the subject to talk about his or her thoughts. Think-aloud strategy gives researchers access to subjects' declarative, procedural, conditional or strategic knowledge (Shaverlson, Ruiz-Primo & Wiley, 2005). In this project, we used the think-aloud strategy to verify and provide validity evidence for the cognitive model for dividing fractions.

Methods

Participants and Setting

Four 5th and 6th grade students were purposefully selected for this study. Students were selected based on (a) the variability of their ability to successfully solve some division of fraction problems, (b) their ability to verbalize their actions, (c) their willingness to participate and (d) the diversity of their ethnic backgrounds, SES status and reported disability. All students attended local public schools. In this report, these four students are identified by their pseudonyms: Alice, Bill, Carl and Diane.

- Alice was a 10-year-old female Asian student; she was identified as a talented and gifted (TAG) student and had skipped kindergarten and second grade while attending a public elementary school. Alice was eligible to receive free and reduced lunch.
- Bill was an 11-year-old male Caucasian student. He had no known disabilities, and he was eligible to receive free and reduced lunch.
- Carl was a 12-year-old male Caucasian student, who was self-identified as having dysgraphia. Carl was not eligible to receive free and reduced lunch.
- Diane was a 12-year-old female Hispanic student. She had no identified disabilities, nor was she eligible for receiving free and reduced lunch.

(See Table 1 for complete demographic characteristics of the participating students.)

The interviews were conducted by the first author at a local university. The one-on-one interviews lasted approximately an hour for each student. Prior to the interviews, the researchers obtained informed consent from the parents and assent from the participating students. The researcher also obtained permission of recording the interviews on videos for the purpose of transcription and data analysis.

Instrument Development

The instrument used in this study included 15 open-ended mathematics items. The items were categorized into three topics: (a) different types of fractions, (b) multiplications of fractions and (c) divisions of fractions. Each item represented one or more attributes of the cognitive model. The association of different cognitive attributes is illustrated in the cognitive model for division of fractions found in Appendix A. In the interview script, the questions about multiplication and divisions of fractions were often proceeded by the same problem types with whole number applications. This arrangement aimed to prime students' thinking and orient their attention to the type of fractions problems they were about to solve. See Appendix B for the items.

The interview script provided three or four prompts for each item. For the items about different types of fractions, the following three questions were posed: "*What do you know about fractions?*" "*What do you think about fractions?*" "*What type of number is it* (when presenting students with written probes)?" This line of questioning was intended to assess students' ability to identify different types of fractions and their conceptual understanding of fractions. For the item about multiplication and division of fractions, the following four questions were posed: "*What type of problem is this*?" "*What is this asking you to do conceptually?*" "*How would you solve this type of problem?*" and "*Does the answer make sense? Why?*" The first question asked students to identify the type of problem (e.g., a multiplication of fractions); the second question asked students to describe their conceptual understanding of the problem; the third question asked students to report the procedure of solving the problem; and the last question asked students to reflect and evaluate their thinking process. These questions, used in conjunction with the items, were designed to encourage students' to articulate their thinking process of solving a computation problem from the perspectives of different depths of knowledge. These items and interview prompts were employed to conduct structured interviews so that students' responses to the items and prompts were comparable.

Procedure of Student Interviews

The first author presented the items, one at a time, to the participating students in the one-on-one interviews. These items were presented in a pre-determined sequence as identified in Appendix B. The students were asked to solve the math problems with paper and pencil, they were not permitted to use calculators in the process. While presenting the student with an item, the first author asked the student open-ended prompts that corresponded to the given item. These questions were used to guide the students to reflect and articulate the steps and strategies they employed to solve the problem. However, based on individual student engagement, the first author scaffolded the questioning and dialogue to encourage the participants to share their thinking processes. She allowed students flexibility in writing down the solutions first and recaptured the process verbally afterwards. She used prompting, paraphrasing the questions, providing additional examples and wait time to optimize the quality of students' responses. She also provided verbal redirect when the students exhibited off-task behaviors, such as going on tangents. To avoid making the students feeling frustrated, the first author stopped prompting students if they did not respond to the scripted questions correctly. This resulted in variable opportunities to respond for each student. The interview process was video recorded for the purpose of data transcription and data analysis. The students' work products of solving the items were collected at the end of interviews.

Data Analysis

Participants' narratives were transcribed verbatim and the content of the items were inserted in the transcripts. Two raters independently reviewed the narratives and evaluated the correspondence between the participants' thought processes and the pre-determined coding scheme. The coding scheme of this study included: (a) the cognitive attributes (see Appendix A) and (b) four levels of cognitive complexity (Appendix C). Based on students' narratives, the raters (a) identified the attribute(s) used, (b) specified the level(s) of cognitive complexity of their procedure or strategies and (c) determined whether the procedure or strategies were used correctly. For example, based on Diane's report of how to convert an improper fraction to a mixed number, the two raters independently consulted the cognitive model and determined that Diane had used Attribute 10 (know and apply the procedure for converting a mixed number to a improper fraction) to solve the problem. Next, both reviewers independently consulted the matrix describing the four level of cognitive complexity (see Appendix C) and determined that Diane not only successfully applied the rules and procedure to solve the problem correctly but also articulated conceptual understanding of converting an improper fraction into a mixed number. Her responses suggested a Level 1 and a Level 2 understanding, respectively. Therefore, in this instance, Diane's narrative of converting an improper fraction to a mixed number was coded as “(a) Attribute 10, (b) Level 1 and Level 2 understanding and (c) correct response.”

To establish consistency of coding, two raters randomly selected one sample case for reviewing together after their independent coding. Discrepancies within the sample case were discussed and reconciled between the two raters. The remaining three transcripts were coded based on the established coding scheme and consensus. After coding, the reviewers consolidated the data and displayed the results in a data matrix.

Results

The data matrix (Table 2) is organized to include the cognitive attributes as rows and the four participants in separate columns. The number for each cognitive attribute in the data matrix was assigned arbitrarily. It was not indicative of the location of the designated attribute in a hierarchical order. Each cell corresponded to the number of occasions that the students answered the questions correctly and incorrectly for each of the four levels of cognitive complexity, for a specific cognitive attribute. The occasions of success were noted as 1; and the conceptual errors, procedural errors or the absence of evidence to indicate success were noted as 0. The X's in the matrix signified that the interviewer stopped asking questions because the student could not provide more information about their understanding (or lack thereof) on the topic. The number of instances of responses to each attribute at different levels of complexity varied from person to person because the number of opportunity to respond differed due to the students' responses and prompts by the interviewer.

Alice answered most questions correctly for Attributes 5, 16, 6, 10, 15, 9, 12, and 3. Her level of understanding for Attributes 5, 16, 6 and 12 was limited in Level 1 understanding. Her level of understanding for Attributes 10, 15, 9 and 3 were extended to Level 2, with some evidence of Level 3 understanding. She did not answer any questions correctly for Attributes 4 and 14. Nor did she use strategies related to Attribute 2 to solve any math problem.

Bill answered most questions correctly for Attributes 16 and 3. He answered some questions correctly for Attributes 6, 10, 15, 12 and 3. For Attributes 16 and 6, Bill demonstrated a Level 1 and a Level 2 understanding. Bill also demonstrated understanding that encompassed Levels 1 to 3 for Attributes 12 and 3. Given five response opportunities, he answered one question correctly for Attribute 5. He did not answer any questions correctly for Attribute 9, 4 and 14, nor did he use strategies related to Attribute 2 to solve the math problems. There were several incorrect responses followed by X's in the coding (Table 2), which suggested that the interviewer stopped asking questions about that particular cognitive attribute and moved on to the next set of questions. This action signified that Bill could not articulate his understanding on the selected cognitive attribute or he clearly demonstrated that he did not understand the concept.

Carl answered most questions correctly for Attributes 2, 5, 16, and 12. He answered some questions correctly for Attributes 6, 10, 15 and 4. Carl demonstrated Level 1 understanding for Attributes 5, 6, 10, 15, and 4; Level 2 understanding for Attributes 5, 15 and 12; and Level 3 understanding for Attributes 10, 15, and 12. Given five opportunities to respond, he answered only one question correctly for Attributes 3. He did not answer any questions correctly for Attribute 14, nor did he use Attribute 9 to solve any math problems. It is noteworthy that there were a series of incorrect responses followed by an X's in Attributes 6, 10, 15, 3 and 14. Carl's lack of understanding of a concept in a basic level did not prevent him from articulating the concept correctly in a higher level (c.f. his responses in Attribute 10 and 15).

Diane answered most questions correctly for Attributes 2, 5, 16, 6, 10, and 3. She answered some questions correctly for Attributes 15, 9, 12, 3 and 4. Diane demonstrated Level 1 understanding for Attributes 2, 5, 16, 6, 10, 15, 12, 3 and 4. She also demonstrated Level 2 understanding for Attributes 2, 5, 6, 10, 9, 12, 3 and 4; and Level 3 understanding for Attributes 16, 10, 15, 9, 12, and 3. She did not answer any questions correctly for Attribute 14.

It is noteworthy that all students answered questions related to Attribute 16 (identify the property of multiplication) correctly; and none of the students answered questions correctly for Attribute 14 (definition of division of fractions). Most instances of success occurred in demonstrating Level 1 understanding; while errors occurred more often in demonstrating Level 2 or Level 3 understanding.

Discussion

The purpose of verbal protocols is to provide insights into students' thinking processes. In this study, we used verbal protocols to better understand the cognitive model underlying mastery of division of fractions. As such, we interviewed four students with varying abilities in the targeted domain to verify the cognitive model developed through previous research efforts. Results will be used to refine and revise the hierarchical organization of the attributes. Although the standardized verbal protocols were used in the interviews, the interviewer did ask additional questions to verify answers or facilitate students' articulation of their understanding of the concepts. The interviewer also exercised professional judgment to move on to the next set of questions and stimuli when the student clearly demonstrated that he or she did not understand the intended concept. As a result, the number of responses varied from student to student, attribute to attribute.

Understanding Students' Thinking

Alice demonstrated proficiency in procedural knowledge of multiplying and dividing with fractions, but she did not understand why it made sense to "flip" the fractions when changing it from division to multiplication. Her limited understanding of these concepts led her to rely on rote memory of the rules. Bill got confused around procedural application of fractions and when each rule was applied. Because he had limited understanding of the concept, he over-relied on what he remembered about the rules, which led to procedural errors. Carl knew the procedure for inverting and multiplying fractions, but he had a very limited conceptual understanding of why it worked. He had a difficult time with many pre-requisite skills such as multiplication of fractions or understanding the definition of fractions. This might suggest there is a level of independence of "invert and multiply" algorithm that does not relate to a deep understanding of fractions, computation or conceptual understanding. Diane had a limited understanding of multiplying fractions; she over-generalized rules stating that multiplying a fraction with another fraction would always lead to a small number. She initially did not recognize inverse property of fractions, but she used inductive reasoning to figure it out.

For all students, procedural and conceptual knowledge of whole number computation does not necessarily help with rational number operations. This was evident with the data related to Attribute 15 (multiplication of fractions). Bill and Carl both committed strategic and factual errors. For example, Carl stated that 9 x 7 = 72. The factual errors led to the wrong answer even though the procedure he applied was correct.

There was considerable variability in student's level of proficiency in using correct math vocabulary in the correct contexts. Students referred to a denominator as a bottom number, and a numerator as a top number. They were not familiar with terms such as improper fractions or mixed numbers. They did not use the term inverse, rather they described the relationship between multiplication and division as "opposite." All students had a very limited or restricted understanding of fractions. They could not put the multiplication or division of fractions in a real-life context other than slicing pizzas. They did not make connections between fractions and proportions, "elements of a set" or "parts to the whole." When it came to the step of "invert and multiply," all students got confused about which number should be inverted.

Verification of the Cognitive Model

For most of the attributes, student responses verified the hierarchical organization. The number of attributes mastered was directly proportional to the mastery of skills: Students with advanced skills in division of fractions demonstrated mastery of more attributes than students with less advanced skills. This evidence corroborates the attribute structure. However, based on the results from this study, several revisions to the cognitive model are warranted:

1. Attribute 4 (limits of division) was moved to reflect an association with Attribute 14 (conceptual understanding of division of fractions). Initially, this attribute was situated between Attribute 6 (definition of fractions) and Attribute 10 (converting improper fractions to mixed numbers and vice versa). The attribute was originally placed between these attributes because of the assumption that in order to master subsequent attributes (such as Attribute 10 and beyond), students must understand that fractions with denominators of zero are undefined. However, in the verbal protocols, only four students demonstrated mastery of Attribute 4. The two students who did not demonstrate mastery successfully responded to prompts related to Attributes 6 and 10. As such, this evidence suggested that the original placement was inaccurate. We placed Attribute 4 prior to Attribute 14 because we anticipated that students with a conceptual understanding of division of fractions will also know the limits of division. The new placement should be evaluated with another subset of students.
2. The direct link between Attribute 6 (definition of fractions) and Attribute 16 (identity property of multiplication) was removed. In the verbal protocols, most of the students responded to questions related to Attribute 16 without any reference to fractions. Because of the relative independence of these attributes, we separated them in the cognitive model.

Results from the verbal protocols could not be used to verify Attribute 3 (relationship between multiplication and division) and Attribute 14 (conceptual understanding of division of fractions). Attribute 3 was originally placed between Attribute 15 (multiplication of fractions) and Attribute 14 (conceptual understanding of division of fractions) because it was assumed that understanding how these operations are related would lead to a deeper understanding of division of fractions. All four students demonstrated mastery of Attribute 3 when related to whole numbers. However, the interviewer did not prompt students to apply these skills to fractions. As such, additional information is needed to verify appropriate placement of this attribute.

Pre-skills needed for Attribute 14 could not be verified because none of the participants in this study demonstrated mastery of this attribute. Additional research with more participants is needed to verify the relative location of Attribute 14 in the cognitive model.

Limitations of the Study

Several limitations of this study may influence the observed results and our interpretation of the findings in relation to the cognitive model. Primarily, this study only included four students. Although care was taken to represent a range of ability levels and select students with different demographic characteristics, the results may not be generalizable to the larger population of students. The limitations of the sample were already observed in that none of the participants had mastered all of the attributes. As such, future studies with larger sample sizes and participants with a broader range of skills are warranted to verify the observed results.

Another limitation of the current study is that no criterion measure was administered to the participants to verify their skills in the targeted domain. A criterion measure would help corroborate the findings of skill mastery observed in this study.

Conclusions

Based on the evidence provided by the verbal protocols, the cognitive model for division of fractions was verified. Minor revisions were made to more precisely reflect the association between attributes.

Table 1.
Participants' Demographic Information in Think-aloud study.

Participants	Alice	Bill	Carl	Diane
Gender	Female	Male	Male	Female
Race	Asian	Caucasian	Caucasian	Hispanic
Grade Level	5^{th} Grade	6^{th} Grade	6^{th} Grade	6^{th} Grade
Eligible for Free or Reduced Lunch	Yes	Yes	No	No
Disability	None	None	Dysgraphia	None

Table 2.
Verbal Protocols Data Display.

Cognitive Attributes of Learning Divisions and Fractions	**Participants**			
	Alice	Bill	Carl	Diane
2*: Understand the an equation should be balanced			L2 (1)	L1 (101) L2 (1)
5: Understand the distributive property	L1 (11)	L1 (1000X) L2 (0X)	L1 (111) L2 (10)	L1 (111) L2 (11)
16: Identity property of multiplication	L1 (111)	L1 (11) L2 (1)	L1 (11)	L1 (111) L2 L3 (1)
6: Understand the definition of fraction	L1 (1111111) L2 (0)	L1 (1101X) L2 (101X)	L1 (1101) L2 (00X)	L1 (11101) L2 (11110111)
10: Know and apply the procedure for converting the mixed number to an improper fraction when a ≠ 0 and b ≠ 0.	L1 (11) L2 (1) L3 (0)	L1 (1011) L2 (10X) L3 (00)	L1 (101111) L2 (000X) L3 (1)	L1 (11111) L2 (11111) L3 (1)
15: Definition of multiplication of fractions	L1 (11111) L2 (0001) L3 (1)	L1 (10000X) L2 (0X) L3 (0)	L1 (000000X) L2 (0011X) L3 (101)	L1 (111111) L2 (001101) L3 (1)
9: Inverse property	L2 (1)	L1 (00X) L2 (0X)	L1 (00X)	L1 (00X) L2 (1) L3 (1)
12: Apply algorithm of invert and multiply (division)	L1 (1111) L2 (0X)	L1 (0100X) L2 (0X)	L1 (11111) L2 (000)	L1 (11111) L2 (10X) L3 (0)
3: Understand the inverse relationship between multiplication and division	L1(11) L2 (1)	L1 (10111) L2 (0X1) L3 (1111)	L1 (00010X) L2 (00X)	L1 (1) L2 (0111001) L3 (110111) L4 (0)
4: Understand the limitation of division	L1 (0X) L2 (00X)	L1 (00X)	L1 (11) L2 (00X)	L1 (11001) L2 (101)
14: Definition of division of fractions	L2 (0X)	L1 (0X) L2 (00X)	L2 (000X) L4 (0X)	L2 (00000X)

1: Correct responses
0: Incorrect responses or no responses
X: Interviewer stopped questioning on the selected concept

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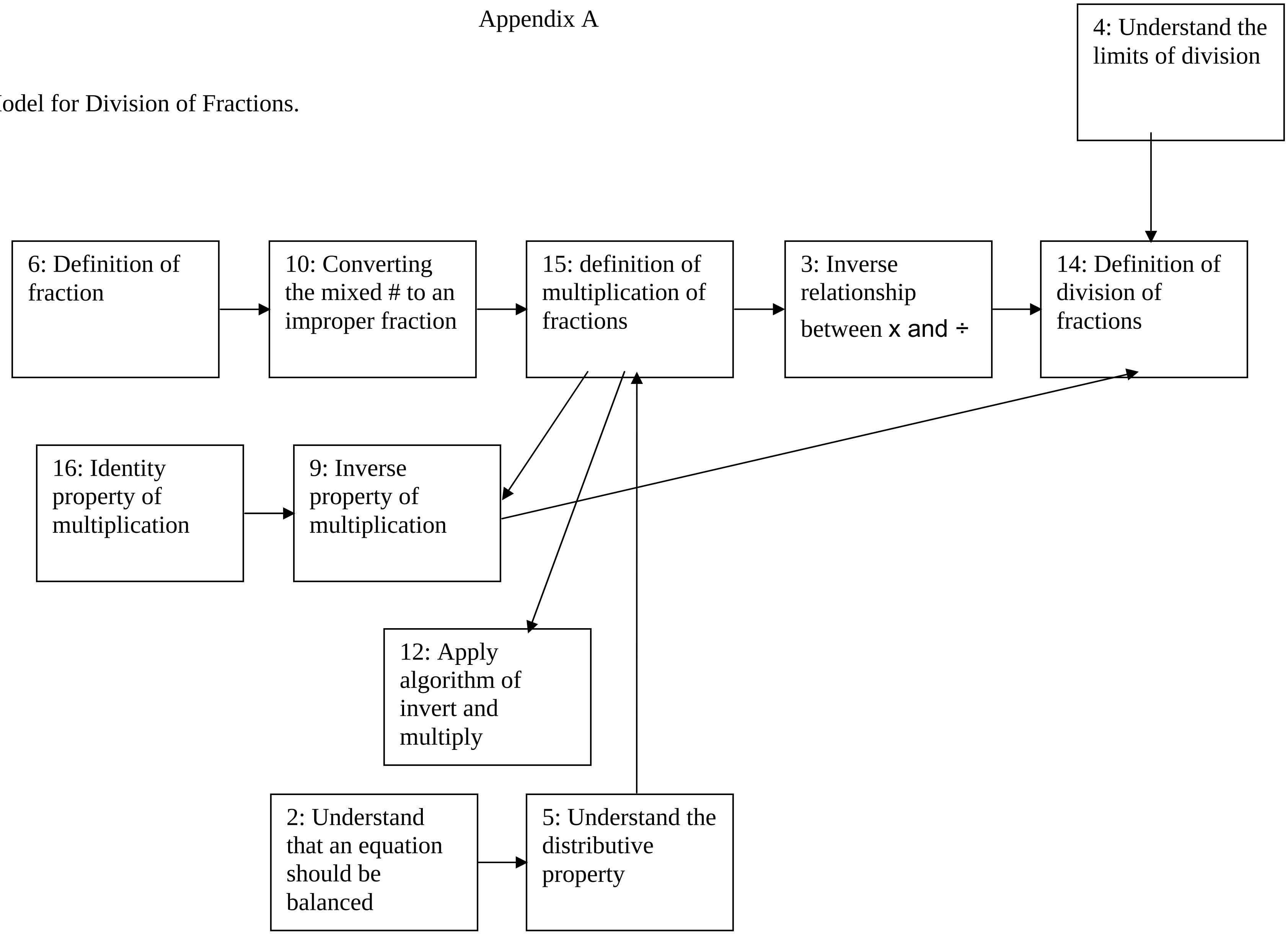
Appendix A

Figure A1.
Cognitive Model for Division of Fractions.



Appendix B

Items

1. Type of Numbers: .67, $\frac{3}{4}$, 92, 36%, $1\frac{2}{5}$

2. $\frac{3}{5}$

3. Type of Fractions: $3\frac{6}{8}$, $\frac{11}{7}$, $\frac{4}{9}$

4. Write another form of this fraction $9\frac{6}{7}$

5. Write another form of this fraction $\frac{32}{5}$

6. 9 x 4

7. Multiplication 12 x $\frac{2}{2}$

8. 6 (2+ 9)

9. $\frac{6}{7} \times \frac{11}{4}$

10. $\frac{3}{4} \times \frac{4}{3}$

11. $2\overline{)12}$

12. $\frac{2}{3} \div \frac{6}{5}$

13. $\frac{7}{11} \div \frac{8}{3}$

14. $\frac{2}{0} \div \frac{1}{2}$

15. If 5x7 = 35, what else do you know?

Appendix C

Table C1.
Depth of Knowledge and Corresponding Examples.

Level	**Depth of Knowledge and Corresponding Examples**
1	Knowledge and application of general facts and simple procedures: • Know vocabulary terms such as divisor, dividend, numerator, denominator; • Label parts of a fraction; • Recognize and differentiate between mixed numbers, fractions, and improper fractions; • Add, subtract, multiply, and divide two numbers
2	Knowledge and application of concepts and complex procedures: • Describe fractions using multiple examples including parts of a whole, elements in a set, or ratio/proportion; • Dissect a mixed number in relation to definition of fractions; • Describe concepts using mathematical language
3	Strategic thinking: • Explain a solution or method of arriving at a solution; • Describe multiple representations of rational numbers such as fractions or decimals; • Selecting the appropriate operation for solving a problem
4	Extended thinking: • Describe concepts using real world examples; • Generalize concepts to new situations